Optimal stochastic control of arrays of wave energy point converters

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Literature


Equations of Motion for an Array of Point Absorbers

- **Point absorber**: Wave energy device with horizontal dimensions small compared to a characteristic wave length. Absorbs energy from all wave directions.
- **Wave gauge**: Sufficient distant from any absorber that no interference takes place.
- **k**: Wave number vector.

Figure 1: Plane irregular wave train impinging on an array of point absorbers.
\textbf{2nd International Summer School on Stochastic Dynamics of Wind Turbines and Wave Energy Absorbers}  
Aalborg, Denmark. 6-8 August, 2014

\begin{itemize}
  \item $m_j$: Structural mass including ballast of absorber $j$
  \item $f_{pj,0}$: Static prestressing force from the power take off system of absorber $j$.
  \item $f_{bj,0}$: Static buoyancy force on absorber $j$.
  \item $f_{cj}(t)$: Dynamic reaction force from the power take off system at the top of $f_{pj,0}$.
  \item $f_{hj}(t)$: Dynamic hydrodynamic force at the top of $f_{bj,0}$.
\end{itemize}

Figure 2: Loads on heave absorber $j$. a) Static equilibrium state. b) Dynamic state.
Absorber $j$ is modeled as a single-degree-of-freedom system defined by the vertical displacement $v_j(t)$ from the static referential state.

**Static equilibrium:**

\[
\begin{align*}
    f_{b,j,0} &= m_j g + f_{p,j,0} \\
    f_{b,j,0} &= \rho D_j(0) g
\end{align*}
\]  

- $D_j(v_j(t))$ : Displaced water volume at displacement $v_j(t)$ from static equilibrium.

**Equation of motion:**

\[
\begin{align*}
    m_j \ddot{v}_j(t) &= f_{h,j}(t) - f_{c,j}(t) , \quad t > t_0 \\
    v_j(t_0) &= v_{j,0} , \quad \dot{v}_j(t_0) = \dot{v}_{j,0}
\end{align*}
\]  

The power take-off force $f_{c,j}(t)$ is positive when acting in opposite direction of $v_j(t)$.

$f_{c,j}(t)$ can be prescribed to control the motion of the absorber. Henceforth, the vector made up of these components will be referred to as the *control force vector*.
Matrix formulation:

\[ m \ddot{\mathbf{v}}(t) = \mathbf{f}_h(t) - \mathbf{f}_c(t) \]  \hspace{1cm} (3)

\[ \mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}, \quad \mathbf{f}_h(t) = \begin{bmatrix} f_{h1}(t) \\ \vdots \\ f_{hn}(t) \end{bmatrix}, \quad \mathbf{f}_c(t) = \begin{bmatrix} f_{c1}(t) \\ \vdots \\ f_{cn}(t) \end{bmatrix} \]  \hspace{1cm} (4)

\[ \mathbf{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \]  \hspace{1cm} (5)

Linear wave theory:

\[ \mathbf{f}_h(t) = \mathbf{f}_b(t) + \mathbf{f}_r(t) + \mathbf{f}_e(t) \]  \hspace{1cm} (6)

- \( \mathbf{f}_h(t) \): Dynamic hydrodynamic force vector.
- \( \mathbf{f}_b(t) \): Dynamic buoyancy force vector.
- \( \mathbf{f}_r(t) \): Radiation force vector.
- \( \mathbf{f}_e(t) \): Wave excitation force vector.

Dynamic buoyancy force vector:

\[ f_b(t) = -\mathbf{r}(\mathbf{v}(t)) = - \begin{bmatrix} r_1(v_1(t)) \\ \vdots \\ r_n(v_n(t)) \end{bmatrix} \]  

(7)

\[ r_j(v_j(t)) = -\rho \left( D_j(v_j(t)) - D_j(0) \right) g \]  

(8)

- \( \mathbf{r}(\mathbf{v}(t)) \) : Bouyancy function. Uncoupled components.

Linearized buoyancy force vector:

\[ r_j(v_j(t)) = -\rho D_j'(0) g v_j(t) = k_j v(t) \quad , \quad k_j = -\rho D_j'(0) g \]  

(9)

\[ f_b(t) = -k\mathbf{v}(t) \]  

(10)

\[ \mathbf{k} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} \]  

(11)
Radiation force vector: \(^1\)

\[
f_r(t) = -m_h \ddot{v}(t) - f_{r,0}(t) 
\]

\[
f_{r,0}(t) = \int_{t_0}^{t} h_{r\dot{v}}(t - \tau)\dot{v}(\tau) \, d\tau 
\]

\[
H_{r\dot{v}}(\omega) = \int_{0}^{\infty} e^{-i\omega t} h_{r\dot{v}}(t) \, dt 
\]

- \(m_h\): Added water mass matrix at infinite large angular frequencies. Full, symmetric, positive definite matrix.
- \(h_{r\dot{v}}(t)\): Causal, symmetric impulse response matrix. \(h_{r\dot{v}}(t) = 0\, , \, t \leq 0\).
- \(H_{r\dot{v}}(\omega)\): Frequency response matrix.

Monochromatic wave excitation:

\[
f_r(t) = -M_h(\omega)\ddot{v}(t) - C_h(\omega)\dot{v}(t)
\]

\[
M_h(\omega) = m_h + \frac{1}{\omega} \text{Im}(H_{r\dot{v}}(\omega)) = m_h - \frac{1}{\omega} \int_0^\infty \sin(\omega t) h_{r\dot{v}}(t) \, dt
\]

\[
C_h(\omega) = \text{Re}(H_{r\dot{v}}(\omega)) = \int_0^\infty \cos(\omega t) h_{r\dot{v}}(t) \, dt
\]

- **\(M_h(\omega)\)**: Hydrodynamic added mass matrix. Symmetric, positive definite.
  \[M_h(\omega) = M_h^T(\omega) = M_h(-\omega)\cdot\]

- **\(C_h(\omega)\)**: Hydrodynamic radiation damping matrix. Symmetric, positive definite.
  \[C_h(\omega) = C_h^T(\omega) = C_h(-\omega)\cdot\]

\[
(m + m_h)\ddot{v}(t) + r(\dot{v}(t)) + \int_{t_0}^t h_{r\dot{v}}(t - \tau)\ddot{v}(\tau) \, d\tau = f_e(t) \quad t > t_0
\]

\[
v(t_0) = v_0, \quad \dot{v}(t_0) = \dot{v}_0
\]

\[\text{(17)}\]
Wave excitation force vector:

\[
f_e(t) = \int_{-\infty}^{\infty} h_{e\eta}(t - \tau)\eta(\tau)\,d\tau
\]

(18)

\[
H_{e\eta}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}h_{e\eta}(t)\,dt
\]

(19)

- \(h_{e\eta}(t)\) : Impulse response vector function. \textit{Not causal}.
- \(H_{e\eta}(\omega)\) : Frequency response vector function.

\(h_{r\dot{v}}(t), H_{r\dot{v}}(\omega), M_h(\omega), C_h(\omega), h_{e\eta}(t), H_{e\eta}(\omega)\) are determined numerically by a linear finite element (FE) or boundary element (BE) program. Increasingly finer mesh required as the frequency is increased. Here is used the BE program WAMIT.\(^1\)

---

Control force vector:

Derivative, Integral, Proportional (DIP) control with acceleration component:

\[
f_c(t) = m_c \ddot{v}(t) + c_c \dot{v}(t) + k_c v(t) + \int_{-\infty}^{\infty} h_{c\dot{v}}(t - \tau) \dot{v}(\tau) \, d\tau
\]  

\begin{itemize}
  \item \( m_c \): Symmetric gain matrix of acceleration component.
  \item \( c_c \): Symmetric gain matrix of velocity (derivative) component.
  \item \( k_c \): Symmetric gain matrix of displacement (proportional) component.
  \item \( h_{c\dot{v}}(t) \): Non-causal weight matrix function (“impulse response function”) related to integral component.
\end{itemize}

It turns out that the control law format in Eq. (20) by a suitable calibration of the gain matrices provides the optimal control of the array of point absorbers in monochromatic as well as irregular 2D or 3D sea-states.

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Optimal Control

The performance index (cost functional) is taken as the mean power take-off (absorbed power) during the control interval $[t_0, t_1]$:

$$\bar{P}_a = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f_c^T(\tau) \dot{v}(\tau) \, d\tau$$

Monochromatic wave excitation. $f_c(t + T) = f_c(t)$, $\dot{v}(t + T) = \dot{v}(t)$:

$$\bar{P}_a = \frac{1}{T} \int_{0}^{T} f_c^T(\tau) \dot{v}(\tau) \, d\tau$$

Stochastic optimal control. Ergodic response processes, infinite control horizon:

$$\bar{P}_a = \lim_{t_1 - t_0 \to \infty} \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f_c^T(\tau) \dot{v}(\tau) \, d\tau = E[f_c^T(t) \dot{V}(t)]$$

- $f_c(t)$ : Control force vector process.
- $\dot{V}(t)$ : Velocity vector process.
Monochromatic Wave Excitation

\[ \eta(t) = \text{Re}(\eta_0 e^{i\omega t}) \]
\[ f_e(t) = \text{Re}(F_e e^{i\omega t}) \quad , \quad F_e = H_e \eta(\omega) \eta_0 \]  

\[ f_b(t) = -kv(t) \quad \text{(linear buoyancy)} \]
\[ f_r(t) = -M_h(\omega)\ddot{v}(t) - C_h(\omega)\dot{v}(t) \quad \text{(Eq. (15))} \]
\[ f_c(t) = m_c\ddot{v}(t) + c_c\dot{v}(t) + k_c v(t) \quad \text{(no integral control)} \]  

\[ M(\omega)\ddot{v}(t) + C(\omega)\dot{v}(t) + K v(t) = f_e(t) = \text{Re}(F_e e^{i\omega t}) \]  

\[ M(\omega) = m + M_h(\omega) + m_c \]
\[ C(\omega) = C_h(\omega) + c_c \]
\[ K = k + k_c \]  

\[ v(t) = \text{Re}(Ve^{i\omega t}) \quad , \quad V = H(\omega) F_e \]  

\[ H(\omega) = (K - \omega^2 M(\omega) + i\omega C(\omega))^{-1} \]  

\[ H(\omega) : \text{Frequency response function of array.} \]
\[
\dot{\mathbf{v}}(t) = \text{Re} \left( i\omega \mathbf{v}_e^{i\omega t} \right) = \text{Re}(i\omega \mathbf{V}) \cos(\omega t) - \text{Im}(i\omega \mathbf{V}) \sin(\omega t)
\]

\[
\mathbf{f}_c(t) = \text{Re} \left( (k_c - \omega^2 \mathbf{m}_c + i\omega \mathbf{c}_c) \mathbf{V}_e^{i\omega t} \right) = \\
\text{Re}( (k_c - \omega^2 \mathbf{m}_c + i\omega \mathbf{c}_c) \mathbf{V}) \cos(\omega t) - \text{Im}( (k_c - \omega^2 \mathbf{m}_c + i\omega \mathbf{c}_c) \mathbf{V}) \sin(\omega t)
\]

Using the symmetry of the gain matrices, \( \mathbf{m}_c = \mathbf{m}_c^T \), \( \mathbf{c}_c = \mathbf{c}_c^T \), \( \mathbf{k}_c = \mathbf{k}_c^T \), the mean power take-off may be calculated as:

\[
\bar{P}_a = \frac{1}{T} \int_0^T \mathbf{f}_c^T(\tau)\dot{\mathbf{v}}(\tau) \, d\tau = \frac{1}{2} \omega^2 \mathbf{V}^T \mathbf{c}_c \mathbf{V}^* 
\]

\[
= \frac{1}{2} \omega^2 \mathbf{F}_e^T \mathbf{H}(\omega) \mathbf{c}_c \mathbf{H}^*(\omega) \mathbf{F}_e^* \quad , \quad \mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M}(\omega) + i\omega \mathbf{C}(\omega))^{-1} 
\]

\( \bar{P}_a \) is optimized with respect to \( \mathbf{m}_c, \mathbf{c}_c, \mathbf{k}_c \). The necessary and sufficient conditions for optimality can be shown to be:

\[
\mathbf{K} - \omega^2 \mathbf{M}(\omega) = 0 \quad \Rightarrow \quad \mathbf{k}_c - \omega^2 \mathbf{m}_c = -\mathbf{k} + \omega^2 (\mathbf{m} + \mathbf{M}_h(\omega)) 
\]

\[
2\mathbf{c}_c = \mathbf{C}(\omega) = \mathbf{C}_h(\omega) + \mathbf{c}_c \quad \Rightarrow \quad \mathbf{c}_c = \mathbf{C}_h(\omega) 
\]

The frequency response matrix at optimal control:

\[ H(\omega) = \frac{1}{i\omega} C^{-1}(\omega) = \frac{1}{2\omega i} C_h^{-1}(\omega) \]  \hspace{1cm} (33)

From Eqs. (31), (32), (33) follows:\(^1\)

\[ \bar{P}_{a,\text{opt}} = \frac{1}{8} F_e^T(\omega)C_h^{-1}(\omega)F_e^*(\omega) = \frac{1}{8} H_{e\eta}^T(\omega)C_h^{-1}(\omega)H_{e\eta}^*(\omega) |\eta_0|^2 \]  \hspace{1cm} (34)

Eqs. (25) and (26) reduce to:

\[ \begin{align*}
    f_e(t) &= 2 C_h(\omega) \ddot{v}(t) \\
    F_e(\omega) &= 2 C_h(\omega) \dot{V}(\omega)
\end{align*} \]  \hspace{1cm} (35)

\[ f_c(t) = m_c \ddot{v}(t) + k_c v(t) + c_c \dot{v}(t) = -(m + M_h(\omega)) \ddot{v}(t) - k \dot{v}(t) + C_h(\omega) \dot{v}(t) \]  \hspace{1cm} (36)

Eqs. (35) and (36) show that at optimal control the control force eliminates the inertial and buoyancy forces, so the wave excitation force depends linearly on the velocity vector. As well, these results turn out to be valid for arrays with non-linear buoyancy and for irregular sea-states.

Irregular Wave Excitation

The results in Eqs. (34) and (36) requires that the linear wave theory is valid, and the hydrodynamic parameters, $h_{r\dot{v}}(t)$, $H_{r\dot{v}}(\omega)$, $M_h(\omega)$, $C_h(\omega)$, $h_{e\eta}(t)$, $H_{e\eta}(\omega)$, can be calculated at all angular frequencies of significance.

This assumption is maintained in the following with the aim to devise a full feed-back control law, valid for 2D or 3D linear, irregular waves and nonlinear buoyancy. The control law is entirely based on measurements or estimation of the response quantities $v_j(t)$, $\dot{v}_j(t)$, $\ddot{v}_j(t)$ of all absorbers.

The following constrained optimal control problem is considered, cf. Eqs. (17), (21):

\[
\max_{f_c(t)} \quad P_a = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f_c^T(\tau) \dot{v}(\tau) d\tau
\]

s.t.
\[
(m + m_h)\ddot{v}(t) + r(v(t)) + \int_{t_0}^{t} h_{r\dot{v}}(t - \tau)\dot{v}(\tau) d\tau = f_e(t) - f_c(t), \quad t > t_0
\]

\[
v(t_0) = v_0, \quad \dot{v}(t_0) = \dot{v}_0
\]
The solution to the problem is based on the property that $h_{r \dot{y}}(t)$ is causal, i.e. $h_{r \dot{y}}(t) = 0$, $t \leq 0$, and hence admits a rational approximation. Then the integro-differential initial value problem can be replaced by a state vector differential equation.

Next, the optimality problem is solved by classical state vector based control theory.

Finally, the rational approximation is eliminated, and the control law is expressed in basic hydrodynamic parameters.

The solution becomes:

$$f_c(t) = - (m + m_h) \ddot{y}(t) - r \left( \dot{y}(t) \right) + \int_t^{t_1} h_{r \dot{y}}(\tau - t) \dot{y}(\tau) \, d\tau \quad (38)$$

The integro-differential equation in Eq. (17) reduces to the vectorial Fredholm integral equation of the 1st kind:

$$\int_{t_0}^{t_1} h_{r \dot{y}}(|t - \tau|) \dot{y}(\tau) \, d\tau = f_c(t) \quad (39)$$

As seen from Eq. (38) the control law for optimal power take-off depends on the present displacement vector $\mathbf{v}(t)$, the present acceleration vector $\ddot{\mathbf{v}}(t)$, and all future velocity vectors $\dot{\mathbf{v}}(\tau), \tau \in [t, t_1]$. The latter need to be predicted based on previous measurements.

The optimal control law eliminates the inertial term $(\mathbf{m} + \mathbf{m}_h)\ddot{\mathbf{v}}(t)$ and the nonlinear buoyancy force vector $\mathbf{r}(\mathbf{v}(t))$ from the integro-differential equation of motion.

Consider the case of infinite control horizon, $t_0 \rightarrow -\infty$, $t_1 \rightarrow \infty$. Then, Fourier transformation of Eq. (39) provides:

$$\int_{-\infty}^{\infty} h_r(|t - \tau|)\dot{\mathbf{v}}(\tau) \, d\tau = \mathbf{f}_e(t) \Rightarrow \left\{ \begin{array}{l} \mathbf{F}_e(\omega) = 2C_h(\omega)\dot{\mathbf{V}}(\omega) \end{array} \right.$$  \hspace{1cm} (40)

Hence, at optimal control in irregular sea-states with infinite control horizon the optimality condition in Eq. (35) derived for monochromatic wave excitation needs to be fulfilled at all significant angular frequencies in the wave excitation.
Power take-off at optimal control:

2D irregular stationary sea-state, infinite control horizon and linear buoyancy, i.e. \( r(\mathbf{v}(t)) = k \mathbf{v}(t) \). The optimal power take-off of the array can be shown to be:

\[
\bar{P}_{a,\text{opt}} = \frac{1}{4} \int_{-\infty}^{\infty} H_{\eta\eta}^T(\omega) C_h^{-1}(\omega) H_{\eta\eta}(\omega) S_{\eta\eta}(\omega) d\omega
\]  

\[ \text{Eq. (41)} \]

- \( S_{\eta\eta}(\omega) \): Double-sided auto-spectral density function of the sea-surface elevation process.

Eq. (41) indicates the generalization of Eq. (34) to plane, irregular sea-states.

Eq. (41) presumes:

- Future velocity vectors \( \mathbf{\dot{v}}(\tau), \tau \in [t, t_1] \) can be predicted without errors.
- Control force vectors of arbitrary magnitude \( |\mathbf{f}_c(t)| \) can be applied (no saturation).
- Displacements \( \mathbf{v}_j(t) \) of arbitrary positive or negative magnitude are allowed.
- First order wave theory is valid.

Prediction Problem

Eq. (39) provides:

\[
\int_{t_0}^{t} h_r \ddot{v}(t - \tau) \dot{v}(\tau) \, d\tau + \int_{t}^{t_1} h_r \ddot{v}(\tau - t) \dot{v}(\tau) \, d\tau = f_e(t) \quad \Rightarrow \\
\int_{t}^{t_1} h_r \ddot{v}(\tau - t) \dot{v}(\tau) \, d\tau = f_e(t) - \int_{t_0}^{t} h_r \ddot{v}(t - \tau) \dot{v}(\tau) \, d\tau \tag{42}
\]

The optimal control law in Eq. (38) attains the form:

\[
f_e(t) = - (m + m_h) \ddot{v}(t) - r(\dot{v}(t)) - \int_{t_0}^{t} h_r \ddot{v}(t - \tau) \dot{v}(\tau) \, d\tau + f_e(t) \tag{43}
\]

Eq. (43) requires the estimation of the wave excitation force vector \( f_e(t) \).

Hence, the velocity vector prediction problem has been changed into a wave load estimation problem. Eq. (43) appears as a mixed feed forward-feed back control law.
Figure 3: Realizations of $v(t)$, $\dot{v}(t)$ and $\ddot{v}(t)$ at optimal control. $H_s = 3$ m, $\gamma = 3.3$. ($T_p = 7.42$ s).  

Realizations of $\mathbf{v}(t)$ under optimal control appear as narrow-banded with the angular centerfrequency equal to the peak-angular frequency $\omega_p$. Then, it is suggested to estimate $\mathbf{f}_e(t)$ from Eq. (35), valid for monochromatic wave excitation:

$$\mathbf{f}_e(t) \simeq 2C_h(\omega_p) \dot{\mathbf{v}}(t)$$  \hspace{1cm} (44)

Correspondingly, the following sub-optimal, causal feed-back control law is obtained:

$$\mathbf{f}_c(t) = -(\mathbf{m} + \mathbf{m}_h) \ddot{\mathbf{v}}(t) - \mathbf{r}(\mathbf{v}(t)) + 2C_h(\omega_p) \dot{\mathbf{v}}(t) - \int_{t_0}^{t} \mathbf{h}_r \ddot{\mathbf{v}}(t - \tau) \dot{\mathbf{v}}(\tau) d\tau$$  \hspace{1cm} (45)

In case of linear buoyancy, $\mathbf{r}(\mathbf{v}(t)) = k \mathbf{v}(t)$, and infinite control horizon the mean power take-off of the array can be shown to be:

$$\bar{P}_a = \frac{1}{4} \int_{-\infty}^{\infty} H^T_{\eta\eta}(\omega) C_h^{-1}(\omega_p) \left( 2C_h(\omega_p) - C_h(\omega) \right) C_h^{-1}(\omega_p) H_{\eta\eta}^*(\omega) S_{\eta\eta}(\omega) d\omega$$  \hspace{1cm} (46)

Example 1: Optimal and Sub-Optimal Control

Figure 4: Geometry of heave absorber.

Table 1: Heave absorber parameters.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>7.00 m</td>
<td>$b$</td>
</tr>
<tr>
<td>$D$</td>
<td>14.00 m</td>
<td>$m$</td>
</tr>
<tr>
<td>$h$</td>
<td>30.00 m</td>
<td>$m_h$</td>
</tr>
</tbody>
</table>
Figure 5: Geometry of a two heave absorber system. Case 1: Attenuator device. Case 2: Terminator device. $\frac{a}{D}$ is varied.
Double-sided JONSWAP auto-spectral density function:¹

\[
S_{\eta\eta}(\omega) = \frac{5}{32} \beta \frac{H_s^2}{\omega_p^5} \left( \frac{|\omega|}{\omega_p} \right)^{-5} \exp \left( -\frac{5}{4} \left( \frac{\omega}{\omega_p} \right)^{-4} \right)
\]

\[
\beta = \left( 1 - 0.287 \ln(\gamma) \right) \gamma^\delta
\]

\[
\delta = \exp \left( -\frac{1}{2} \left( \frac{|\omega| - \omega_p}{\sigma \omega_p} \right)^2 \right)
\]

\[
\sigma = \begin{cases} 
0.07 & , \quad |\omega| \leq \omega_p \\
0.09 & , \quad |\omega| > \omega_p 
\end{cases}
\]

\[
T_p = \frac{2\pi}{\omega_p} = \sqrt{\frac{180 H_s}{g}}
\]

- \(T_p\) : Peak period. Peak angular frequency (modal frequency): \(\omega_p = \frac{2\pi}{T_p}\).
- \(H_s\) : Significant wave height. Standard deviation: \(\sigma_\eta = \frac{1}{4} H_s\).
- \(\gamma\) : Band-width parameter.

Figure 6 One-sided JONSWAP auto-spectral density function.
a) As a function of the bandwidth parameter $\gamma$, $H_s = 3$ m.
b) As a function of the significant wave height $H_s$, $\gamma = 3.3$
Figure 7: Mean power take-off of the array as a function of $\gamma$, $H_a = 3.0$ m.
Thick lines: Case 1. Thin lines: Case 2. - - : Optimal control. - - - : Sub-optimal causal control.
a) a/D=6.0. b) a/D=4.5. c) a/D=3.0. d) a/D=1.5.
Figure 8: Interaction factor for mean power take-off of the array as a function of $\gamma$, $H_s = 3.0$ m. Thick lines: Case 1. Thin lines: Case 2. —: Optimal control. - - -: Sub-optimal causal control. a) a/D=6.0. b) a/D=4.5. c) a/D=3.0. d) a/D=1.5.
Figure 9: Mean power take-off of the array as a function of $H_s$, $\gamma = 3.3$.
Thick lines: Case 1. Thin lines: Case 2. -- : Optimal control. ---: Sub-optimal causal control.
a) a/D=6.0. b) a/D=4.5. c) a/D=3.0. d) a/D=1.5.
Figure 10: Interaction factor for the mean power take-off as a function of $H_s$, $\gamma = 3.3$.
Thick lines: Case 1. thin lines: Case 2. - - -: Optimal control. - -: Sub-optimal causal control.
a) $a/D=6.0$. b) $a/D=4.5$. c) $a/D=3.0$. d) $a/D=1.5$. 
Constrained Control

- Hot spot stresses caused by control force $f_c(t)$. Linear structural response: $\sigma_j(t) = b_j f_c(t)$.

$$v_{\min} \leq v(t) \leq v_{\max}$$

(49)

$$f_{c,\min} \leq f_c(t) \leq f_{c,\max}$$

(50)

Figure 12: Dynamic buoyancy force with artificial control stiffness close to the constraints.

\[ f_b(t) = -r(v) = -k \cdot \begin{cases} 
- \frac{0.50}{16.00 + v} & , \quad v \in ] - 16.00, -15.75] \\
-2 & , \quad v \in ] - 15.75, -2.00] \\
v & , \quad v \in ] - 2.00, 7.00] \\
v + \frac{(v - 7)^3}{147} & , \quad v \in ]7.00, 13.75] \\
2.9145 & , \quad v \in ]13.75, 14.00[ \\
\frac{14.00 - v}{14.00} & , \quad v \in [14.00, \infty] 
\end{cases} \]
\[ f_c(t) = \begin{cases} f_{c,\max} & , \quad f_{c,0}(t) \geq f_{c,\max} \\ f_{c,0}(t) & , \quad f_{c,\min} < f_{c,0}(t) < f_{c,\max} \\ f_{c,\min} & , \quad f_{c,0}(t) \leq f_{c,\min} \end{cases} \] (52)

- \( f_{c,0}(t) \) : Unconstrained control force.

Figure 13: Constrained control force.
Unconstrained control force (causal, sub-optimal):

\[ f_{c,0}(t) = -(m + m_h)\ddot{v}(t) - r(v(t)) + k_c v(t) + 2c_c \dot{v}(t) - \int_{t_0}^{t} h_{r\ddot{v}}(t-\tau) \dot{v}(\tau) d\tau \]

(53)

- \( k_c, c_c \) : Free gain factors. Determined so the constrained power take-off becomes maximum.

\( k_c = 0 \), linear buoyancy, \( r(v(t)) = kv(t) \), and infinite control horizon. The optimal derivative gain factor is given as, cf. Eq. (45):\(^1\)

\[ c_0 = c_{c,\text{opt}} = \frac{\int_{-\infty}^{\infty} C_h(\omega) S_{F_e F_e}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{F_e F_e}(\omega) d\omega} \left( \simeq C_h(\omega_p) \right) \]

(54)

The solution method will be Monte Carlo simulation. We shall follow the determination of the responses \( \dot{v}(t), \ddot{v}(t) \) during the intervals \([t_1, t_2], [t_2, t_3], [t_3, t_4]\) on Figure 13.

---

$[t_1, t_2[, f_c(t) = f_{c,\text{max}}$. Solution of the integro-differential equation, cf. Eq. (17):

$$(m + m_h)\ddot{v}(t) + r(v(t)) + \int_{t_1}^{t} h(t-\tau)\dot{v}(\tau)\,d\tau = f_c(t) - f_{c,\text{max}} \quad t > t_1$$

$v(t_1) = v_1 \quad \dot{v}(t_1) = \dot{v}_1$

(55)

$[t_2, t_3[, f_c(t) = f_{c,0}(t)$. Numerical solution of the differential equations:

$$\begin{align*}
  k_c v(t) + 2c_c \dot{v}(t) &= f_c(t) \quad t > t_2 \\
  k_c \ddot{v}(t) + 2c_c \dot{v}(t) &= \dot{f}_c(t)
\end{align*}$$

$v(t_2) = v_2 \quad \dot{v}(t_2) = \dot{v}_2$

(56)

$[t_3, t_4[, f_c(t) = f_{c,\text{min}}$. Numerical solution of the integro-differential equation:

$$(m + m_h)\ddot{v}(t) + r(v(t)) + \int_{t_1}^{t} h(t-\tau)\dot{v}(\tau)\,d\tau = f_c(t) - f_{c,\text{min}} \quad t > t_3$$

$v(t_3) = v_3 \quad \dot{v}(t_3) = \dot{v}_3$

(57)
Random phase model of wave excitation force:\(^1\)

\[
\begin{align*}
    f_e(t) &= \sum_{j=1}^{J} \sqrt{2} f_j \cos(\omega_j t - \Phi_j) \\
    \dot{f}_e(t) &= -\sum_{j=1}^{J} \sqrt{2} f_j \omega_j \sin(\omega_j t - \Phi_j)
\end{align*}
\]  

\[
f_j = \sqrt{2 |H_{\eta}(\omega_j)|^2 S_{\eta\eta}(\omega_j) \Delta \omega}
\]  

- \(\Phi_j\): Mutual independent, identical distributed random variables. Uniformly distributed in \([0, 2\pi]\).
- \(\Delta \omega\): Bandwidth of frequency discretization, \(\omega_j = j \Delta \omega\).

Mean power take-off by ergodic sampling:\(^1\)

\[
\bar{P}_a = \bar{P}_a(c_c, k_c) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f_c(t) \dot{v}(t) \, dt
\]  

Positive loop. Power is transferred from the ocean to the generator.

Negative loop. Power is transferred from the generator to the ocean.
Generated electric power:\(^{1,2}\)

\[
\begin{align*}
   P_e^+ (t) &= \eta^+ P_a^+ (t) \\
   P_a^- (t) &= \eta^- P_e^- (t)
\end{align*}
\]

\(\eta^+\): Efficiency coefficient related to positive power take-off loops.
\(\eta^-\): Efficiency coefficient related to negative power take-off loops.

\[
P_e (t) = P_e^+ (t) + P_e^- (t) = \eta^+ P_a^+ (t) + \frac{1}{\eta^-} P_a^- (t) = \\
\left( \eta^+ H(f_c(t)\dot{v}(t)) + \frac{1}{\eta^-} \left( 1 - H(f_c(t)\dot{v}(t)) \right) \right) f_c(t)\dot{v}(t)
\]

\(H(x) = \begin{cases} 
1 & , \ x \geq 0 \\
0 & , \ x < 0 
\end{cases}\)
Example 2: Constrained Optimal Control of a Single Point Absorber

Table 2: Heave absorber parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>7.00 m</td>
</tr>
<tr>
<td>$D$</td>
<td>14.00 m</td>
</tr>
<tr>
<td>$h$</td>
<td>30.00 m</td>
</tr>
<tr>
<td>$v_{\text{max}}$</td>
<td>14.00 m</td>
</tr>
<tr>
<td>$v_{\text{min}}$</td>
<td>-16.00 m</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0.25 m</td>
</tr>
<tr>
<td>$H_s$</td>
<td>3.0 m</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>3.3</td>
</tr>
<tr>
<td>$b$</td>
<td>8.00 m</td>
</tr>
<tr>
<td>$m$</td>
<td>$1.84 \cdot 10^6$ kg</td>
</tr>
<tr>
<td>$m_h$</td>
<td>$0.44 \cdot 10^6$ kg</td>
</tr>
<tr>
<td>$\eta^+$</td>
<td>0.75</td>
</tr>
<tr>
<td>$\eta^-$</td>
<td>0.75</td>
</tr>
<tr>
<td>$c_0$</td>
<td>$1.00 \cdot 10^5$ Ns/m</td>
</tr>
</tbody>
</table>

$f_{c,\text{max}} = -f_{c,\text{min}}$ is varied in the examples.
Figure 15: Mean power outtake as a function of $c_c$, $k_c = 0$. 
Figure 16: Mean power output as a function of $k_c$, $c_c = c_0$. 

- Unconstrained
- $f_{c,\text{max}} = 1 \times 10^6$
- $f_{c,\text{max}} = 0.8 \times 10^6$
- $f_{c,\text{max}} = 0.6 \times 10^6$
- $f_{c,\text{max}} = 0.4 \times 10^6$
- $f_{c,\text{max}} = 0.2 \times 10^6$
- $f_{c,\text{max}} = 0.1 \times 10^6$
Figure 17: Mean electrical power take-off as a function of $c_e$, $k_c = 0$. 
Figure 18: Mean electrical power take-off as a function of $k_c$, $c_c = 2.156c_0$. 
Figure 19: Performance of sub-optimal controllers as a function of $c_c$, $k_c = 0.5k$, $f_{c,\text{max}} = \infty$.

a) Mean absorbed power. b) Mean electrical power take-off.

--- $f_{c,0}(t)$, Eq. (53). -- $f_{c,0}(t) = c_c \dot{v}(t)$. --- $f_{c,0}(t) = c_c \dot{v}(t) + k_c v(t)$. 
What’s Next

- Better prediction of future velocity vectors using nonlinear stochastic filtering theory.¹

- Theoretical based optimal control of constrained point absorbers. *Hamilton-Jacobi-Bellman* equation in combination to *Pontryagin’s maximum principle*. The so-called "dimensional curse" may limit this approach to a SDOF system.²

- The coupling between the two problems.

- Non-linear wave theory. Stochastic linearization?

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